Existence of a Universal Deformation Ring (Mazur)

Recall p prime G topological group

G profinite $\Longleftrightarrow$ $G \cong \varprojlim_{\alpha} G_\alpha$ – finite discrete top

G pro-p $\Longleftrightarrow$ $G \cong \varprojlim_{\alpha} G_\alpha$ — $|G_\alpha| = p^{r_\alpha}$

Examples

$\hat{\mathbb{Z}} = \varprojlim_n \mathbb{Z}/n\mathbb{Z}$

$\mathbb{Z}_p^* = \varprojlim_n (\mathbb{Z}/p^n\mathbb{Z})^*$

$\cup\!\mathrm{l}$ profinite but not pro-p.

$\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

$\mathbb{Z}_p$

$GL_n(\mathbb{Z}_p)$ not pro-p

$\cup\!\mathrm{l}$

$I_1 = \{A \in GL_n(\mathbb{Z}_p) \mid A \equiv \mathrm{id} \bmod p\}$

§1 Fininess condition Φ_p

Def G profinite. We say that G satisfies Φ_p if $\forall\, G_0 \subseteq G$ open finite index, the following equiv conditions hold.

① maximal pro-p quotient of G_0 is topologically fin gen.

(2) $\mathrm{Hom}(G_0, \mathbb{F}_p)$ is a fin. dim $\mathbb{F}_p$ - VS

Burnside's basis thm

Examples

- $K/\mathbb{Q}_p$ finite. $\mathrm{Gal}(\overline{K}/K)$ satisfies $\Phi_q \ \forall q$

- $F/\mathbb{Q}$ finite $S \subseteq F$ fin set of places

  $F_S \subseteq \overline{F}$ max. ~~ext~~. unram outside of S

  then $\mathrm{Gal}(F_S/F)$ satisfies $\Phi_p \ \forall p$.

Idea: can check on G^{ab}. pass to $\mathrm{Gal}(K^{ab}/K)$

$$p^{\alpha}\mathbb{Z}_p \times \cdots \times p^{\gamma}\mathbb{Z}_p \times \prod_{l \neq p} l^{r}\mathbb{Z}_l$$

exercise.

§2 Deformation

p prime $\quad \mathbb{F}$ finite field of char p

$W(\mathbb{F})$ ring of Witt vectors

$W(\mathbb{F}_q) = \mathcal{O}_K$, $K/\mathbb{Q}_p$ finite unram.

G profinite group.

$V_{\mathbb{F}}$ fin dim continuous G-rep over $\mathbb{F}$

discrete top

$\mathcal{UR}_{W(\mathbb{F})}$ = cat of all finite local Artinian $W(\mathbb{F})$-alg. with residue field $\mathbb{F}$.

Def $A \in \mathcal{UR}_{W(\mathbb{F})}$

① deformation of $V_{\mathbb{F}}$ to A is a pair (V_A, ι)

- V_A free A-module w/ cont. G-action
- $\iota : V_A \otimes_A \mathbb{F} \xrightarrow{\sim} V_{\mathbb{F}}$ G-equiv. iso of $\mathbb{F}$-VS

② framed deformation of $(V_{\mathbb{F}}, \beta)$ to A. (V_A, ι, β_A)

- V_A

— ι

— β_A is an A-basis of V_A with $\iota(\beta_A) = \beta$.

<u>Def</u> deformation functors

① $D_{V_{\mathbb{F}}} : \mathcal{CR}_{W(\mathbb{F})} \longrightarrow \underline{\text{Sets}}$

$$D_{V_{\mathbb{F}}}(A) = \{\text{isom classes of } (V_A, \iota)\}$$

② $D^{\square}_{V_{\mathbb{F}}} : \mathcal{CR}_{W(\mathbb{F})} \longrightarrow \underline{\text{Sets}}$

$$D^{\square}_{V_{\mathbb{F}}}(A) = \{\text{isom classes of } (V_A, \iota, \beta_A)\}$$

<u>Rmk</u> $V_{\mathbb{F}}$ is of the form $\overline{\rho} : G \to GL_n(\mathbb{F})$

$$D_{\overline{\rho}}(A) = D^{\square}_{\overline{\rho}}(A) \big/ \ker(GL_n(A) \longrightarrow GL_n(\mathbb{F}))$$

§3 (pro-) representability

p prime G profinite.

<u>Prop</u> If G satisfies Φ_p

① $D^{\square}_{V_{\mathbb{F}}}$ is pro-representable by a complete local Noetherian $W(\mathbb{F})$-alg. $R^{\square}_{V_{\mathbb{F}}}$

($\Leftrightarrow$ $\exists$ isom, functorial in A

$$D^{\square}_{V_{\mathbb{F}}}(A) \xrightarrow{\sim} \mathrm{Hom}_{W(\mathbb{F})}(R^{\square}_{V_{\mathbb{F}}}, A).$$

② If $\mathrm{End}_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, then $D_{V_{\mathbb{F}}}$ — " — $R_{V_{\mathbb{F}}}$.

Def $R_{V_{\mathbb{F}}}$ universal deformation ring

$R^{\square}_{V_{\mathbb{F}}}$ ... framed deformation ring.

Proof ① $\dim_{\mathbb{F}} V_{\mathbb{F}} = n$. say $\bar{\rho} : G \to GL_n(\mathbb{F})$

G profinite $\Rightarrow$ $G = \varprojlim_{\alpha} \underbrace{G/H_\alpha}_{\text{finite}}$

have a group presentation

$$\langle g_1, \dots, g_s \mid r_1, \dots, r_t \rangle$$

Define $R := W(\mathbb{F})[X_k^{ij} \mid k = 1, \dots, s,\ i,j = 1, \dots, n] / I$

where $I = (r_1(x_1, \dots, x_s), \dots, r_t(x_1, \dots, x_s))$

Take $J := \ker(R \longrightarrow \mathrm{Mat}_{n\times n}(\mathbb{F}))$

$$X_k \longmapsto \bar{\rho}(g_k)$$

$$R_\alpha^\square := \widehat{R}^J$$

has a unique rep.

$$\rho_\alpha^\square : G/H_\alpha \longrightarrow GL_n(R_\alpha^\square)$$

$$g_k \longmapsto X_k$$

$(R_\alpha^\square, \rho_\alpha^\square)$ universal :

$\forall \rho_A : G \longrightarrow GL_n(A)$, $\rho_A^\alpha : G/H_\alpha \longrightarrow GL_n(A)$

$\exists! \ \phi : R_\alpha^\square \longrightarrow A$ s.t. $\rho_A^\alpha = \phi \circ \rho_\alpha^\square$

$$X_k^{ij} \longmapsto \text{entries of } \rho_A^\alpha(g_k)$$

Take $(R^{\square}_{V_{\mathbb{F}}}, \rho^{\square}_{V_{\mathbb{F}}}) := \varprojlim_{\alpha} (R^{\square}_{\alpha}, \rho^{\square}_{\alpha})$

satisfies U.P. and $R^{\square}_{V_{\mathbb{F}}}$ is complete local.

NTS $R^{\square}_{V_{\mathbb{F}}}$ is Noetherian $\Leftrightarrow$ $m_R/(m_R^2, p)$ is a fin dim

$\mathbb{F}$-VS. $\square$

§4 Tangent Spaces

$\mathbb{F}[\varepsilon] = \mathbb{F}[X]/(X^2)$ ring of dual numbers over $\mathbb{F}$

Def Zariski tangent space for $D_{V_{\mathbb{F}}}$ is $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$

—— // —— $D^{\square}_{V_{\mathbb{F}}}$ is $D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$

Rmk By representability we have iso of $\mathbb{F}$-VS

$$D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) = \mathrm{Hom}_{W(\mathbb{F})}(R_{V_{\mathbb{F}}}, \mathbb{F}[\varepsilon]) \xrightarrow{\sim} \mathrm{Hom}_{\mathbb{F}}(m_R/(m_R^2, p), \mathbb{F})$$

$$\| $$

$$t_{R_{V_{\mathbb{F}}}}$$

tangent space.

Lemma ① $\exists$ canonical isom.

$$D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) \xrightarrow{\sim} H^1(G, \operatorname{ad} V_{\mathbb{F}})$$

$\operatorname{ad} V_{\mathbb{F}} = \operatorname{End}_{\mathbb{F}}(V_{\mathbb{F}})$

② If G satisfies Φ_p then $D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$ is a finite dim. $\mathbb{F}$-VS

③ $\dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) + n^2 - h^0(G, \operatorname{ad} V_{\mathbb{F}})$

Proof ① we use $\operatorname{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \xrightarrow{\sim} H^1(G, \operatorname{ad} V_{\mathbb{F}})$

Take $V_{\mathbb{F}[\varepsilon]} \in D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$

$$\begin{array}{ccccccccc} 0 & \to & \varepsilon V_{\mathbb{F}[\varepsilon]} & \to & V_{\mathbb{F}[\varepsilon]} & \to & V_{\mathbb{F}} & \to & 0 \\ & & \| & & \varepsilon \downarrow 1 & & & & \\ & & V_{\mathbb{F}} & & & & & & \end{array}$$

so $V_{\mathbb{F}[\varepsilon]} \in \operatorname{Ext}^1_{\mathbb{F}[G]}(V_{\mathbb{F}}, V_{\mathbb{F}}) \xrightarrow{\sim} H^1(G, \operatorname{ad} V_{\mathbb{F}})$

Conversely if we have $\mathbb{F}[G]$-equivariant

$$0 \to V_{\mathbb{F}} \to E \to V_{\mathbb{F}} \to 0$$

$\wr\wr \quad \nearrow$

$\xi V_{\mathbb{F}[\xi]}$

E is an $\mathbb{F}[\xi]$-module w/ cont G-action and reduces to $V_{\mathbb{F}}$ so $E \in D_{V_{\mathbb{F}}}(\mathbb{F}[\xi])$

(2) G satisfies Φ_p. fix $\bar{\rho} : G \to GL(V_{\mathbb{F}})$

Take $G' = \ker(\bar{\rho})$ open. finite index in G.

Let $(V_{\mathbb{F}[\xi]}, \rho_{\mathbb{F}[\xi]}) \in D_{V_{\mathbb{F}}}(\mathbb{F}[\xi])$, $r : GL(V_{\mathbb{F}[\xi]}) \to GL(V_{\mathbb{F}})$

$\bar{\rho} = r \circ \rho_{\mathbb{F}[\xi]}$. know that $\ker(r)$ is a pro-p group

$\Rightarrow G'/(\ker \rho|_{G'})$ is a pro-p group.

have G'/H maximal pro-p group $\quad \Phi_p \Rightarrow$

G/H top. fin. gen.

then all $\rho_{\mathbb{F}[\xi]}$ factorize via G/H

$\Rightarrow \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\xi]) < \infty$.

(3) fix a basis of $V_{\mathbb{F}}$. $V_{\mathbb{F}[\varepsilon]} \in D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon])$

$\dim_{\mathbb{F}} \{\text{bases of } V_{\mathbb{F}[\varepsilon]} \text{ lifting } V_{\mathbb{F}}\} = n^2$.

Take β, β' bases of $V_{\mathbb{F}[\varepsilon]}$ compare them on

$$0 \to \varepsilon V_{\mathbb{F}[\varepsilon]} \to V_{\mathbb{F}[\varepsilon]} \to V_{\mathbb{F}} \to 0.$$

$$\iota : (V_{\mathbb{F}[\varepsilon]}, \beta) \longrightarrow (V_{\mathbb{F}[\varepsilon]}, \beta') \quad \text{a map}$$

$$\iota \bmod \varepsilon = id$$

$\iota(\beta) \cong \beta'$ G-equivariant $\leftrightarrow$ elements of $(\operatorname{ad} V_{\mathbb{F}})^G$.

$h^0(G, \operatorname{ad} V_{\mathbb{F}})$

$$\Rightarrow \dim_{\mathbb{F}} D^{\square}_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) = \dim_{\mathbb{F}} D_{V_{\mathbb{F}}}(\mathbb{F}[\varepsilon]) + n^2 - h^0(G, \operatorname{ad} V_{\mathbb{F}})$$

$\square$

<u>Cor</u> $R_{V_{\mathbb{F}}}$ and $R^{\square}_{V_{\mathbb{F}}}$ are Noetherian.

§5 Traces

Thm (Mazur '87. Carayol '91)

$V_{\mathbb{F}}$ absolutely irreducible rep of profinite gp G. $V_A, V_A' \in D_{V_{\mathbb{F}}}(A)$ s.t.

$$tr(\sigma | V_A) = tr(\sigma | V_A'), \ \forall \sigma \in G$$

then $V_A \cong V_A'$.

§6 Representability of $D_{V_{\mathbb{F}}}$

Prop $End_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$. (e.g. $V_{\mathbb{F}}$ abs. irred).

$\Rightarrow D_{V_{\mathbb{F}}}$ is representable.

Def Define $D_{V_{\mathbb{F}}}$ as follows.

① Objects: $\forall A \in \mathcal{U}R_{W(\mathbb{F})}$, $D_{V_{\mathbb{F}}}(A)$ is a category

obj. (V_A, ι_A)

mor. A-linear isom $(V_A, \iota) \xrightarrow{\sim} (V_A', \iota')$

compatible w/ G-action.

(2) morphisms: $\forall A \to A'$ morphism in $\mathfrak{U}R_{W(\mathbb{F})}$,

have a cover.

$(V_A, L_A) \longrightarrow (V_{A'}, L_{A'})$ consists of an

equivalence class

$$[\alpha] : V_A \otimes_A A' \xrightarrow{\sim} V_{A'}$$

A'-linear isom compatible w/ G-action

$[\alpha]$, $[\alpha']$ are equivalent if they differ by an element in $A'^{\times}$.

<u>Def</u> $\widehat{\mathfrak{U}R_{W(\mathbb{F})}}$:= cat of complete local Noetherian $W(\mathbb{F})$-alg.

$$\left(\widehat{\mathfrak{U}R_{W(\mathbb{F})}}\right)^{op} \cong \text{cat of formal spectrum of c. l. N. } W(\mathbb{F})\text{-alg}.$$

<u>Def</u> An equivalence relation $R \rightrightarrows X$

in $(\widehat{\mathcal{U}\mathcal{R}_{\text{sup}}})^{op}$ is defined by two morphism $R \to X$,

$R \to X$, s.t.

① the induced $R \longrightarrow X \times X$ is a closed embedding

② $\forall T \in \widehat{\mathcal{U}\mathcal{R}}_{W(\mathbb{F})}$, $R(T) \subseteq X \times X(T)$ is an equivalence relation on sets.

<u>Recall</u>

have $W(\mathbb{F})$-group scheme PGL_n

e counit map σ section of e

$id = \sigma \circ \sigma$

$\ker(id) \subseteq PGL_n$ closed subscheme

$\widehat{PGL_n} = \widehat{PGL_n}^{I}$

$\in (\widehat{\mathcal{U}\mathcal{R}}_{W(\mathbb{F})})^{op}$.

- $X_{V_{\mathbb{F}}} := \operatorname{Spf} R^{\square}_{V_{\mathbb{F}}}$

<u>Proof</u> $\forall A \in \mathcal{U}R_{W(\mathbb{F})}$, $\widehat{PGL_n}(A) \circlearrowright V_A$ by conjugation.

$\Rightarrow \widehat{PGL_n} \circlearrowright X_{V_{\mathbb{F}}} \Rightarrow$ gives an equiv relation in $(\widehat{\mathcal{U}R}_{W(\mathbb{F})})^{op}$

$$X_{V_{\mathbb{F}}} \times \widehat{PGL_n} \rightrightarrows X_{V_{\mathbb{F}}}$$

$$(x, g) \longmapsto (x, gx)$$

Indeed. $End_{\mathbb{F}[G]}(V_{\mathbb{F}}) = \mathbb{F}$, $\widehat{PGL_n} \circlearrowright X_{V_{\mathbb{F}}}$ freely.

have closed embedding

$$X_{V_{\mathbb{F}}} \times \widehat{PGL_n} \longrightarrow X_{V_{\mathbb{F}}} \times X_{V_{\mathbb{F}}}$$

$$(x, g) \longmapsto (x, gx)$$

why can we have the quotient $X_{V_{\mathbb{F}}}/\widehat{PGL_n}$?

<u>Thm</u> [SGA3, VI b. Thm 14]

$R \underset{d_1}{\overset{d_0}{\rightrightarrows}} X_{V_{\mathbb{F}}}$ equivalence relation in $(\widehat{\mathcal{U}R}_{W(\mathbb{F})})^{op}$ s.t.

d_0 is flat then $X_{V_{\mathbb{F}}}/R$ exists in $(\widehat{\mathcal{U}R}_{W(\mathbb{F})})^{op}$.

check axioms to see

$$\mathrm{Spf}\, R_{V_{\mathbb{F}}} := \widehat{X_{V_{\mathbb{F}}} / PGL_n}$$ is universal

$\square$